APPROXIMATE CALCULATION OF THE POTENTIAL FLOW OF A FLUID JET FLOWING IN A THIN LAYER OVER THE SURFACE OF A SOLID BODY

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The flows of the type indicated in the title may take place at the point of impact of a liquid jet upon a rock, in the buckets of Pelton-type hydraulic turbines, etc.

We shall introduce curvilinear coordinates x_1, x_2 on the surface of a body, where the coordinate lines are to be directed along the lines of main curvature.

The third coordinate

$$x_3 = h \tag{1}$$

is equal to the length of the normal measured from the surface of a solid body.

Then we obtain a system of orthogonal coordinates. A linear element here is equal to

$$ds = \sqrt{H_1^2 dx_1^2 + H_2^2 dx_2^2 + H_3^2 dx_3^2} = \sqrt{H_1^2 dx_1^2 + H_2^2 dx_2^2 + dh^2} \tag{2}$$

Let the free surface of the flow be given by the equation

$$h = h^\circ (x_1, x_2) \tag{3}$$

The quantity h° is taken to be small of the first order. On this surface the pressure is atmospheric and, consequently, the stream velocity is constant. Consequently, on the free surface we have

$$\frac{1}{H_1^2}\left(\frac{\partial\varphi}{\partial x_1}\right)^2 + \frac{1}{H_2^2}\left(\frac{\partial\varphi}{\partial x_2}\right)^2 + \frac{1}{H_3^2}\left(\frac{\partial\varphi}{\partial x_3}\right)^2 = u_0^2 \tag{4}$$

On the solid surface ($x_3 = 0$) we have

$$\partial\varphi / \partial x_3 = 0 \tag{5}$$

therefore, the last term on the right-hand side of Equation (4) is small, it is of the second order and may be omitted. Therefore, we have

$$\frac{1}{H_1^2}\left(\frac{\partial\varphi}{\partial x_1}\right)^2 + \frac{1}{H_2^2}\left(\frac{\partial\varphi}{\partial x_2}\right)^2 = w_0^2 \tag{6}$$

Let us consider further the limiting case of zero thickness of the fluid layer. In this case we have a limiting value

$$\varphi = \varphi_0(x_1, x_2) \tag{7}$$

This function is determined from the first-order equation

$$\frac{1}{H_1^2}\left(\frac{\partial\varphi_0}{\partial x_1}\right)^2 + \frac{1}{H_2^2}\left(\frac{\partial\varphi_0}{\partial x_2}\right)^2 = w_0^2 \tag{8}$$

which can be solved by known methods (see [1], Chap. 9, 5). This solution may be extended to apply to the region of flow close to the surface on the basis of the equation

$$\frac{\partial}{\partial x_1}\left(\frac{H_2H_3}{H_1}\frac{\partial\varphi_0}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{H_3H_1}{H_2}\frac{\partial\varphi_0}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{H_1H_2}{H_3}\frac{\partial\varphi_0}{\partial x_3}\right) = 0 \tag{9}$$

and the initial condition

$$\partial\varphi_0 / \partial x_3 = 0 \qquad \text{for } x_3 = 0 \tag{10}$$

Then the function ϕ is presented in the form

$$\varphi = \varphi_0 + \varphi_1 \tag{(11)}$$

where ϕ_1 is small of the first order. Substituting expression (11) into (6) and using Equation (8), we obtain

$$\frac{2}{H_1^2}\frac{\partial\varphi_0}{\partial x_1}\frac{\partial\varphi_1}{\partial x_1} + \frac{2}{H_2^2}\frac{\partial\varphi_0}{\partial x_2}\frac{\partial\varphi_1}{\partial x_2} + h^\circ\left[\frac{\partial}{\partial x_3}\left(\frac{1}{H_1^2}\right)\left(\frac{\partial\varphi_0}{\partial x_1}\right)^2 + \frac{\partial}{\partial x_3}\left(\frac{1}{H_2^2}\right)\left(\frac{\partial\varphi_0}{\partial x_2}\right)^2\right] = 0$$

or

$$\frac{1}{H_1^2}\frac{\partial\varphi_0}{\partial x_1}\frac{\partial\varphi_1}{\partial x_1} + \frac{1}{H_2^2}\frac{\partial\varphi_0}{\partial x_2}\frac{\partial\varphi_1}{\partial x_2} + h^\circ\left[\frac{\varkappa_1}{H_1^2}\left(\frac{\partial\varphi_0}{\partial x_1}\right)^2 + \frac{\varkappa_2}{H_2^2}\left(\frac{\partial\varphi_0}{\partial x_2}\right)^2\right] = 0 \tag{12}$$

where κ_1, κ_2 are the main curvatures of the solid surface.

The thickness of the layer h° (x , x_2) is obtained from the equation of continuity.

Let *dn* be the distance between two streamlines, indefinitely closely adjacent on the solid surface, *dV* the volume rate of flow between the stream surfaces formed in first approximation by the normals to the surface intersecting with these streamlines. Then we have

$$h^{\circ} w dn = dV, \quad \text{or} \quad h^{\circ} = \frac{1}{w}\frac{dV}{dn}$$

Then, up to an error of the second order, the velocity w may be replaced by the constant value of zero approximation w_0 and the streamline may be determined by zero approximation, i.e. on the basis of Equation (8). And so we have

$$h^{\circ}(x_1, x_2) = \frac{1}{w_0}\frac{dV}{dn}$$

This expression is substituted in Equation (12); thereupon it is transformed into a first-order equation and solved by known methods (see [1], Chap. 3, 3).

To calculate the pressure on the surface of the body, a solution of Equation (12) is not necessary, because in the first approximation

$$p - p_0 = \rho h^{\circ}\left[\frac{\varkappa_1}{H_1^2}\left(\frac{\partial \varphi_0}{\partial x_1}\right)^2 + \frac{\varkappa_2}{H_2^2}\left(\frac{\partial \varphi_0}{\partial x_2}\right)^2\right] \tag{13}$$

where p_0 is atmospheric pressure κ_1 and κ_2 are the main curvatures. It should be remembered that h° is small of the first order and, therefore, when taking into account ϕ_1, a correction is obtained which is small of second order.

BIBLIOGRAPHY

1. Stepanov, V.V., *Kurs differentsial'nykh uravnenii (Course in Differential Equations)*. GTTI, 1953.

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